

A Note on Bounding Regret of the C²UCB Contextual Combinatorial Bandit

Bastian Oetomo Malinga Perera Renata Borovica-Gajic Benjamin I. P. Rubinstein*

February 21, 2019

Abstract

We revisit the proof by Qin et al. (2014) of bounded regret of the C²UCB contextual combinatorial bandit. We demonstrate an error in the proof of volumetric expansion of the moment matrix, used in upper bounding a function of context vector norms. We prove a relaxed inequality that yields the originally-stated regret bound.

1 Introduction

In deriving a regret bound on the C²UCB contextual combinatorial bandit, Qin et al. (2014) use the following equality within the proof of their Lemma 4.2.

Claim 1. Let k, m, n be natural numbers, \mathbf{V} be a $d \times d$ real and positive definite matrix, and $S_t \subseteq [m]$ with $|S_t| \leq k \leq m$ for $t \in [n]$. Let $\mathbf{x}_t(i) \in \mathbb{R}^d$ be vectors for $t \in [n], i \in [m]$, and define $\mathbf{V}_n = \mathbf{V} + \sum_{t=1}^n \sum_{i \in S_t} \mathbf{x}_t(i) \mathbf{x}_t(i)^T$. Then $\det(\mathbf{V}_n) = \det(\mathbf{V}) \prod_{t=1}^n \left(1 + \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}}^2 \right)$, where we define $\|\mathbf{a}\|_{\mathbf{M}} = \sqrt{\mathbf{a}^T \mathbf{M} \mathbf{a}}$.

We present a counterexample to Claim 1 in Section 2, and then in Section 3 prove the relaxation given by,

Lemma 2. Under the same conditions as Claim 1, $\det(\mathbf{V}_n) \geq \det(\mathbf{V}) \prod_{t=1}^n \left(1 + \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}}^2 \right)$.

In the setting of C²UCB, $[n], [m]$ correspond to rounds and arms, S_t the (super arm of) played arms in round t , $x_t(i)$ the context vector for arm i at round t , and V_t the covariance matrix from the played contexts added to V (taken to be a scaled identity, for achieving ridge regression reward estimates). We detail in Section 4 how Lemma 2 can be used within the remainder of the proof of (Qin et al., 2014, Lemma 4.2), ultimately yielding the C²UCB regret bound originally claimed. The regret analysis of C²UCB is based on previous analysis of contextual bandits (Auer, 2002; Dani et al., 2008; Chu et al., 2011). We demonstrate that the bound in Lemma 2 is sharp, by describing conditions for equality.

Notation. We denote by $\lambda_i(\mathbf{A})$ the eigenvalues of the $n \times n$ matrix \mathbf{A} , where, without loss of generality, $\lambda_1(\mathbf{A}) \leq \lambda_2(\mathbf{A}) \leq \dots \leq \lambda_n(\mathbf{A})$. We likewise order $S_t = \{s_{(1,t)}, s_{(2,t)}, \dots, s_{(|S_t|,t)}\}$, where $s_{(1,t)} < s_{(2,t)} < \dots < s_{(|S_t|,t)}$.

Generalised Matrix Determinant Lemma. We make use of the identity: Let \mathbf{A} be an invertible $n \times n$ matrix, and \mathbf{B}, \mathbf{C} be $n \times m$ matrices, then $\det(\mathbf{A} + \mathbf{B}\mathbf{C}^T) = \det(\mathbf{I}_m + \mathbf{C}^T \mathbf{A}^{-1} \mathbf{B}) \det(\mathbf{A})$.

2 A Counterexample

Claim 1 derives from the assertion within the proof of (Qin et al., 2014, Lemma 4.2) that,

$$\det(\mathbf{V}_{n-1}) \det \left(\mathbf{I} + \sum_{i \in S_n} (\mathbf{V}_{n-1}^{-1/2} \mathbf{x}_n(i)) (\mathbf{V}_{n-1}^{-1/2} \mathbf{x}_n(i))^T \right) = \det(\mathbf{V}_{n-1}) \det \left(\mathbf{I} + \sum_{i \in S_n} \|\mathbf{x}_n(i)\|_{\mathbf{V}_{n-1}}^2 \right).$$

This appears to conflate outer and inner products, after basis transformation by $\mathbf{V}_{n-1}^{-1/2}$. The following counterexample to Claim 1 establishes that indeed it does not hold in general.

*School of Computing and Information Systems, University of Melbourne, Parkville, VIC 3010, Australia. {boetomo, wpperera}@student.unimelb.edu.au, {rborovica, brubinstein}@unimelb.edu.au

Example 3. Consider $n = 1$, the 2×2 matrix $\mathbf{V} = 1.2\mathbf{I}_2$, $S_t = \{1, 2, 3\}$ and let $\mathbf{x}_1(1) = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$, $\mathbf{x}_1(2) = \begin{bmatrix} 0.6 \\ 0.1 \end{bmatrix}$, $\mathbf{x}_1(3) = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$. It follows that $\mathbf{V}_1 = \begin{bmatrix} 1.66 & 0.32 \\ 0.32 & 1.95 \end{bmatrix}$. Then we have

$$\begin{aligned}
& \det(\mathbf{V}) \prod_{t=1}^n \left(1 + \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}^{-1}}^2 \right) \\
&= \det(\mathbf{V}) \left(1 + \sum_{i=1}^3 \mathbf{x}_1(i)^T \mathbf{V}^{-1} \mathbf{x}_1(i) \right) \\
&= \det(1.2\mathbf{I}_2) \left(1 + \mathbf{x}_1(1)^T \left(\frac{1}{1.2} \mathbf{I} \right) \mathbf{x}_1(1) + \mathbf{x}_1(2)^T \left(\frac{1}{1.2} \mathbf{I} \right) \mathbf{x}_1(2) + \mathbf{x}_1(3)^T \left(\frac{1}{1.2} \mathbf{I} \right) \mathbf{x}_1(3) \right) \\
&= 1.2^2 \left(1 + \frac{1}{1.2} (0.3^2 + 0.7^2) + \frac{1}{1.2} (0.6^2 + 0.1^2) + \frac{1}{1.2} (0.1^2 + 0.5^2) \right) \\
&= 2.892 \neq 3.1346 = 1.66 \times 1.95 - 0.32 \times 0.32 = \det(\mathbf{V}_1).
\end{aligned}$$

3 Proof of Lemma 2

Let $\mathbf{X}_n = [\mathbf{x}_n(s_{(1,n)}) \ \dots \ \mathbf{x}_n(s_{(|S_n|,n)})]$. Then,

$$\begin{aligned}
\det(\mathbf{V}_n) &= \det \left(\mathbf{V} + \sum_{t=1}^n \sum_{i \in S_t} \mathbf{x}_t(i) \mathbf{x}_t(i)^T \right) \\
&= \det \left(\mathbf{V} + \sum_{t=1}^{n-1} \sum_{i \in S_t} \mathbf{x}_t(i) \mathbf{x}_t(i)^T + \sum_{i \in S_n} \mathbf{x}_n(i) \mathbf{x}_n(i)^T \right) \\
&= \det(\mathbf{V}_{n-1} + \mathbf{X}_n \mathbf{X}_n^T) \\
&= \det(\mathbf{V}_{n-1}) \det(\mathbf{I}_{|S_n|} + \mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n) \\
&= \det(\mathbf{V}_{n-1}) \left[\prod_{i=1}^{|S_n|} \lambda_i(\mathbf{I}_{|S_n|} + \mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n) \right] \\
&= \det(\mathbf{V}_{n-1}) \left[\prod_{i=1}^{|S_n|} (1 + \lambda_i(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n)) \right],
\end{aligned}$$

where the fourth and final equalities follow from the Generalised Matrix Determinant Lemma and the fact that adding the identity to a square matrix increases eigenvalues by one. Now, the final line's product can be expanded as

$$1 + \sum_{i=1}^{|S_n|} \lambda_i(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n) + \sum_{1 \leq i_1 < i_2 \leq |S_n|} \lambda_{i_1}(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n) \lambda_{i_2}(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n) + \dots + \prod_{i=1}^{|S_n|} \lambda_i(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n). \quad (1)$$

Since \mathbf{V} is positive definite and $\mathbf{x}_t(i) \mathbf{x}_t(i)^T$ is positive semi-definite (with one eigenvalue being $\mathbf{x}_t(i)^T \mathbf{x}_t(i)$ and the remainder all zero) for all t and i , we have that $\mathbf{V}_{n-1} = \mathbf{V} + \sum_{t=1}^{n-1} \sum_{i \in S_t} \mathbf{x}_t(i) \mathbf{x}_t(i)^T$ is positive definite. Therefore, we conclude that \mathbf{V}_{n-1}^{-1} is also positive definite, hence it has a symmetric square root matrix $\mathbf{V}_{n-1}^{-1/2}$. It also follows that $\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n$ is positive semi-definite. Therefore, the terms starting from the third term in the expansion (1) are all non-negative because they are products of the eigenvalues of $\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n$. Thus we have,

$$\begin{aligned}
\det(\mathbf{V}_n) &= \det(\mathbf{V}_{n-1}) \left[\prod_{i=1}^{|S_n|} (1 + \lambda_i(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n)) \right] \\
&\geq \det(\mathbf{V}_{n-1}) \left(1 + \sum_{i=1}^{|S_n|} \lambda_i(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n) \right) \\
&= \det(\mathbf{V}_{n-1}) (1 + \text{tr}(\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n)) \\
&= \det(\mathbf{V}_{n-1}) \left(1 + \sum_{i \in S_n} \mathbf{x}_n(i)^T \mathbf{V}_{n-1}^{-1} \mathbf{x}_n(i) \right)
\end{aligned}$$

$$= \det(\mathbf{V}_{n-1}) \left(1 + \sum_{i \in S_n} \|\mathbf{x}_n(i)\|_{\mathbf{V}_{n-1}^{-1}}^2 \right),$$

where the third equality follows from expanding out the argument to the trace as

$$\mathbf{X}_n^T \mathbf{V}_{n-1}^{-1} \mathbf{X}_n = \begin{bmatrix} \mathbf{x}_n(s_{(1,n)})^T \mathbf{V}_{n-1}^{-1} \mathbf{x}_n(s_{(1,n)}) & \cdots & \mathbf{x}_n(s_{(1,n)})^T \mathbf{V}_{n-1}^{-1} \mathbf{x}_n(s_{(|S_n|,n)}) \\ \vdots & \ddots & \vdots \\ \mathbf{x}_n(s_{(|S_n|,n)})^T \mathbf{V}_{n-1}^{-1} \mathbf{x}_n(s_{(1,n)}) & \cdots & \mathbf{x}_n(s_{(|S_n|,n)})^T \mathbf{V}_{n-1}^{-1} \mathbf{x}_n(s_{(|S_n|,n)}) \end{bmatrix}.$$

Applying our recurrence relation on \mathbf{V}_t for $1 \leq t \leq n$, we can telescope to arrive at the result.

4 Implication of Lemma 2

By rearranging the inequality, we know that

$$\prod_{t=1}^n \left(1 + \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}^{-1}}^2 \right) \leq \frac{\det(\mathbf{V}_n)}{\det(\mathbf{V})},$$

provided that $\det(\mathbf{V}) > 0$, which is guaranteed for our positive definite \mathbf{V} . The next steps of (Qin et al., 2014, Lemma 4.2)'s proof follow the original pattern¹ now with the second inequality in what follows (due to our Lemma 2 and monotonicity), rather than the original equality:

$$\begin{aligned} \sum_{t=1}^n \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}^{-1}}^2 &\leq 2 \sum_{t=1}^n \log \left(1 + \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}^{-1}}^2 \right) = 2 \log \left[\prod_{t=1}^n \left(1 + \sum_{i \in S_t} \|\mathbf{x}_t(i)\|_{\mathbf{V}_{t-1}^{-1}}^2 \right) \right] \\ &\leq 2 \log \left(\frac{\det(\mathbf{V}_n)}{\det(\mathbf{V})} \right) = 2 \log(\det(\mathbf{V}_n)) - 2 \log(\det(\mathbf{V})), \end{aligned}$$

which yields the regret bound as presented by Qin et al. (2014), without further modification to the proof of their Lemma 4.2.

5 Discussion

The proof of Lemma 2 offers intuition as to when the inequality holds with equality. Namely, it is true when the matrix $\mathbf{X}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{X}_t$ has at most one non-zero eigenvalue *i.e.*, be either a rank-1 or rank-0 matrix for all $1 \leq t \leq n$. This is because the terms that we dropped in calculating the determinant of $\mathbf{I}_{|S_t|} + \mathbf{X}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{X}_t$ are then identically 0. This agrees with the result of the non-generalised matrix determinant lemma.

This occurs when intra-round, played context vectors are co-linear to each other: if the context vector of arm i can be written as $\mathbf{x}_t(i) = a_{it} \mathbf{u}_t$, then we can write $\mathbf{X}_t = \mathbf{u}_t \mathbf{a}_t^T$, where \mathbf{a}_t is a column vector with a_{it} as its components. The matrix we are interested in becomes $\mathbf{X}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{X}_t = (\mathbf{u}_t \mathbf{a}_t^T)^T \mathbf{V}_{t-1}^{-1} (\mathbf{u}_t \mathbf{a}_t^T) = \|\mathbf{u}_t\|_{\mathbf{V}_{t-1}^{-1}}^2 \mathbf{a}_t \mathbf{a}_t^T$, which is a rank-1 matrix. Thus, it also follows that the trace of this matrix is $\|\mathbf{u}_t\|_{\mathbf{V}_{t-1}^{-1}}^2 \|\mathbf{a}_t\|^2$. One interesting thing to notice here is that the context vectors need not to be co-linear across rounds.

A special case of the co-linearity scenario is the non-combinatorial bandit. In this scenario, $|S_t| = 1$ for all t . This means that given a particular round t , there is only one context vector available. In particular, $\det(\mathbf{I}_{|S_t|} + \mathbf{X}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{X}_t) = \det(\mathbf{I}_1 + \mathbf{x}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{x}_t) = 1 + \mathbf{x}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{x}_t$, which is the bound that we had for calculating $\det(\mathbf{I}_{|S_t|} + \mathbf{X}_t^T \mathbf{V}_{t-1}^{-1} \mathbf{X}_t)$, were $|S_t| = 1$.

References

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¹Here as in the original proof, we leverage assumptions: $\lambda_1(V) \geq k$ and the context vectors are of bounded norm $\|\mathbf{x}_t(i)\|_2 \leq 1$.